On Heisenberg's Uncertainty Principle and the CCR

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Realizing the canonical commutation relations (CCR) $[N, \Theta] = -i$ as $N = -i\,\mathrm{d}/\mathrm{d}\vartheta$ and Θ to be the multiplication by ϑ on the Hilbert space of square integrable functions on $[0, 2\pi]$, in the physical literature there seems to be some contradictions concerning the Heisenberg uncertainty principle $\langle \Delta N \rangle \langle \Delta \Theta \rangle \geq 1/4$. The difficulties may be overcome by a rigorous mathematical analysis of the domain of state vectors, for which Heisenberg's inequality is valid. It is shown that the exponentials $\exp\{itN\}$ and $\exp\{is\Theta\}$ satisfy some commutation relations, which are not the Weyl relations. Finally, the present work aims at a better understanding of the phase and number operators in non-Fock representations.

1. Introduction

There is a great variety of papers, which stress the difficulties, contradictions, and the existence of number and phase operators, N resp. Θ, satisfying the canonical commutation relations $[N, \Theta] = -i$ (see [1], [2], and references therein).

Many examples of number and phase operators are based on a representation of N as differential operator $N = -i\dfrac{\mathrm{d}}{\mathrm{d}\vartheta}$ and Θ to be the multiplication operator with ϑ acting on the Hilbert space of square integrable function on the interval $[0, 2\pi]$ (cf. e.g. [2], [3], [4], etc.).

Taking an eigenstate of N we have $\langle \Delta N \rangle = 0$, which seems to be a contradiction to Heisenberg's uncertainty relation $\langle \Delta N \rangle \langle \Delta \Theta \rangle \geq 1/4$, which arises from the CCR $[N, \Theta] = -i$ (cf. e.g. [2]).

The misunderstanding lies in the fact that the operator N is unbounded, and therefore the uncertainty relations are only valid for those state vectors, which are elements of the domain of definition of the commutator $[N, \Theta]$.

Here we strengthen a rigorous mathematical analysis of the domain of definition for operators in a Hilbert space $\mathscr{H}$ to overcome the above difficulty. For an extension of Heisenberg's uncertainty principle to be valid for more state vectors than those in the domain of the commutator $[A, B]$, we define the weak commutator of the selfadjoint observables A and B. Both formulations of the uncertainty relations agree on all vectors of $\mathscr{H}$, if A and B are both bounded operators on $\mathscr{H}$. However, if the selfadjoint A and B satisfy the CCR, they cannot both be bounded (see p. 274 in [5] and Theorem 2 below).

According to the difference and the link of commutator and weak commutator, in Sect. 3 we introduce the weak version of the CCR, which implies the usual one. Two selfadjoint operators A and B satisfying the weak CCR both have to be unbounded (Theorem 2). E.g., the momentum and position operators, $P = -i\dfrac{\mathrm{d}}{\mathrm{d}x}$ resp. Q (multiplication by x), on the Hilbert space $\mathsf{L}^2(\mathbb{R})$ fulfill the weak CCR. When the weak CCR are valid one does not realize the difference, if for the Heisenberg uncertainty relations state vectors f are taken, which are not in the domain of $[A, B]$ (resp. $[P, Q]$).

In the above case of N on $\mathsf{L}^2\left([0, 2\pi], \dfrac{\mathrm{d}\vartheta}{2\pi}\right)$ we first have to make N to become a selfadjoint operator. Contrary to P on $\mathsf{L}^2(\mathbb{R})$ there is an infinity of selfadjoint extensions N_α of N, which are indicated by $\alpha \in [0, 1[$, a well-known result [5]. N_α and Θ fulfill the usual CCR but not its weak version, a result which is connected with the boundedness of Θ (each of the N_α is unbounded). In the weak commutator of N_α and Θ there occurs an additional term which transfers to the weakly formulated Heisenberg uncertainty relations. With these relations, which now are really valid for all state vectors in the domain of N_α, the above mentioned difficulties disappear.

We now turn to an investigation of the commutation relations of the exponentials. If the selfadjoint A and B (more exactly, the associated unitary groups) satisfy the Weyl relations

$$\exp\{itA\}\exp\{isB\} = \exp\{its\}\exp\{isB\}\exp\{itA\}; \quad \forall s, t \in \mathbb{R}, \quad (1)$$

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then A and B obey the weak CCR, and consequently both are unbounded. E.g., it is well known that the above P and Q fulfill (1) (cf. [5], [6]). This contrasts the case of the bounded Θ and the unbounded N_α, $\alpha \in [0, 1[$. Here we obtain modified "Weyl relations", which are expressed by an additional shift in the index α: For every $\alpha \in [0, 1[$ we have

$$\exp\{itN_\alpha\}\exp\{is\Theta\} = \exp\{its\}\exp\{is\Theta\}\exp\{itN_{(\alpha-s)|\mathrm{mod}\,1}\}; \quad \forall s, t \in \mathbb{R}.$$

(See [5] for some more examples, where the CCR do not lead to the Weyl relations.)

In Sect. 5 we finally apply the commutation relations of the exponentials of Θ and the N_α, $\alpha \in [0, 1[$, (Theorem 4) to obtain number operators N_α^L with spectrum $\mathbb{Z} + \alpha$ and a unitary observable phase operator U_L, so that each of the N_α^L implements unitarily the gauge transformations on the Weyl algebra, and $\exp\{itN_\alpha^L\}\, U_L \exp\{-itN_\alpha^L\} = \exp\{-it\}\, U_L$; $\forall t \in \mathbb{R}$, that is in the sense of [3]. Here we use the GNS representation Π_L of the gauge-invariant, macroscopic fully coherent state ω_L on the bosonic C*-Weyl algebra, which associates uniquely to the arbitrary but unbounded linear form L on the one-boson testfunction space E. This setup belongs to a boson system in the thermodynamic limit, where a classical, macroscopic field part appears, and is only possible for infinite dimensional E. For finite-dimensional one-boson spaces the number operators are bounded from below, and hence no phase operator exists (for more details, see [3]).

However, for the one-mode problem it is a great progress to get a phase operator Φ on the one-mode Fock space which fulfills the CCR $[a^* a, \Phi] = -i$ in some classical limit [7], where here the number operator $a^* a$ has positive spectrum.

2. Preliminaries, Heisenberg's Uncertainty Relations

For a better survey and a better understanding let us recapitulate the basic concepts of domain and of selfadjointness for bounded resp. unbounded operators (cf. e.g. [5], [8], [6]). Selfadjointness is fundamental for a physical observable since its spectrum is real and one has the spectral calculus. Especially, Stone's theorem is only valid for selfadjoint operators.

Let $\mathscr{H}$ be a complex Hilbert space with right-linear scalar product $\langle . | . \rangle$. A linear operator A on $\mathscr{H}$ is defined on a domain $\mathscr{D}(A)$, which is a complex subspace of $\mathscr{H}$, and with image in $\mathscr{H}$. A is said to be densely defined if $\mathscr{D}(A)$ is dense in $\mathscr{H}$.

The domain $\mathscr{D}(A^*)$ of the adjoint A^* of a densely defined A is given by those vectors $h \in \mathscr{H}$ for which there exists a $g_h \in \mathscr{H}$ with $\langle h | Af \rangle = \langle g_h | f \rangle$; $\forall f \in \mathscr{D}(A)$. $\mathscr{D}(A^*)$ is a complex subspace of $\mathscr{H}$ and $A^* h := g_h$; $\forall h \in \mathscr{D}(A^*)$. We mention, it may happen that $\mathscr{D}(A^*) = \{0\}$.

A densely defined A on $\mathscr{H}$ is called *symmetric*, if $A \subseteq A^*$, that is, if $\mathscr{D}(A) \subseteq \mathscr{D}(A^*)$ and $Af = A^* f$; $\forall f \in \mathscr{D}(A)$, or equivalently, if $\langle Af | g \rangle = \langle f | Ag \rangle$ for any $f, g \in \mathscr{D}(A)$, which implies $\langle f | Af \rangle \in \mathbb{R}$; $\forall f \in \mathscr{D}(A)$.

A is said to be *selfadjoint* if and only if A is symmetric with $\mathscr{D}(A) = \mathscr{D}(A^*)$. A symmetric A on $\mathscr{H}$ is defined to be *essentially selfadjoint* if it has a unique extension to a selfadjoint $\bar{A}$ on $\mathscr{H}$, that is $\bar{A} = \bar{A}^*$ and $Af = \bar{A} f$; $\forall f \in \mathscr{D}(A) \subseteq \mathscr{D}(\bar{A})$.

The domain $\mathscr{D}(AB)$ of the product of the linear operators A and B on $\mathscr{H}$ is given by those $f \in \mathscr{D}(B)$ for which $Bf \in \mathscr{D}(A)$. Further, $\mathscr{D}(A + B) = \mathscr{D}(A) \cap \mathscr{D}(B)$. Then

$$ABf = A(Bf); \quad \forall f \in \mathscr{D}(AB),$$

$$(A + B)f = Af + Bf; \quad \forall f \in \mathscr{D}(A + B).$$

Hence the domain of the commutator $[A, B] := AB - BA$ is given as $\mathscr{D}([A, B]) = \mathscr{D}(AB) \cap \mathscr{D}(BA)$. Since for unbounded A or unbounded B the domain $\mathscr{D}([A, B])$ in general is a proper subspace of $\mathscr{D}(A) \cap \mathscr{D}(B)$, for selfadjoint A and B on $\mathscr{H}$ we define the *weak commutator* as

$$\langle Af | Bg \rangle - \langle Bf | Ag \rangle; \quad \forall f, g \in \mathscr{D}(A) \cap \mathscr{D}(B), \quad (2)$$

which obviously agrees with $\langle f | [A, B]\, g \rangle$ for $f \in \mathscr{D}(A) \cap \mathscr{D}(B)$ and $g \in \mathscr{D}([A, B])$, and thus gives an extension of the usual commutator $[A, B]$.

If A and B are bounded selfadjoint operators on $\mathscr{H}$, then they are defined on all of $\mathscr{H}$, that is, $\mathscr{D}(A) = \mathscr{D}(B) = \mathscr{H}$. In this case the weak commutator (2) agrees with $\langle f | [A, B]\, g \rangle$ for any $f, g \in \mathscr{H}$.

A similar problem concerning the domain is the definition of the variance $\mathrm{Var}(A, f)$ of the selfadjoint A with respect to the state vector $f \in \mathscr{D}(A)$. Here we define

$$\mathrm{Var}(A, f) := \| (A - \langle f | Af \rangle \mathbb{1}) f \|^2 = \| Af \|^2 - \langle f | Af \rangle^2, \quad f \in \mathscr{D}(A). \quad (3)$$

Usually $\mathrm{Var}(A, f)$ is defined to be $\langle f | (A - \langle f | Af \rangle \mathbb{1})^2 f \rangle$, which makes only sense for $f \in \mathscr{D}(A^2)$, and which obviously for $f \in \mathcal{D}(A^2)$ agrees with (3). However, $\mathcal{D}(A^2) \subseteq \mathcal{D}(A)$, and thus (3) is the more general definition of the variance.

Using the above considerations concerning the domains let us take a look on Heisenberg's uncertainty principle for the selfadjoint A and B on $\mathcal{H}$, where now the weak commutator from (2) comes into play.

Proposition 1 (Heisenberg's uncertainty principle). *Let A and B be selfadjoint operators on the Hilbert space $\mathcal{H}$. It follows*

$$\mathrm{Var}(A, f)\,\mathrm{Var}(B, f) \geq \tfrac{1}{4}|\langle Af\,|\,Bf\rangle - \langle Bf\,|\,Af\rangle|^2; \quad \forall f \in \mathcal{D}(A) \cap \mathcal{D}(B). \quad (4)$$

Proof: Set $A' := A - \langle f\,|\,Af\rangle \mathbb{1}$ and $B' := B - \langle f\,|\,Bf\rangle \mathbb{1}$. Then $\mathcal{D}(A') = \mathcal{D}(A)$ and $\mathcal{D}(B') = \mathcal{D}(B)$, and for the weak commutators we have

$$\langle A'f\,|\,B'g\rangle - \langle B'f\,|\,A'g\rangle = \langle Af\,|\,Bg\rangle - \langle Bf\,|\,Ag\rangle; \quad \forall f, g \in \mathcal{D}(A) \cap \mathcal{D}(B).$$

Thus with the Cauchy-Schwarz inequality one obtains

$$\begin{aligned}&\tfrac{1}{2}|\langle Af\,|\,Bf\rangle - \langle Bf\,|\,Af\rangle| \\ &= \tfrac{1}{2}|\langle A'f\,|\,B'f\rangle - \langle B'f\,|\,A'f\rangle| \\ &= |\mathrm{Re}\,\langle A'f\,|\,B'f\rangle| \leq |\langle A'f\,|\,B'f\rangle| \leq \|A'f\|\;\|B'f\|\end{aligned}$$

for all $f \in \mathcal{D}(A) \cap \mathcal{D}(B)$. □

Assuming $f \in \mathcal{D}([A, B])$, eq. (4) implies

$$\mathrm{Var}(A, f)\,\mathrm{Var}(B, f) \geq \tfrac{1}{4}\,|\langle f\,|\,[A, B]\,f\rangle|^2; \quad \forall f \in \mathcal{D}([A, B]), \quad (5)$$

which is the form of the Heisenberg uncertainty relations usually treated in the literature. If A or B is unbounded, then $\mathcal{D}([A, B])$ in general is a proper subspace of $\mathcal{D}(A) \cap \mathcal{D}(B)$. Hence in this case the version (4) of the uncertainty principle, which uses the weak commutator, is more general than (5). Even, if A and B fulfill the CCR, in general it is not possible to come back from (5) to (4), cf. Section 4. (If one wants to come back from (5) to (4), one needs a sequence $\{f_n; n \in \mathbb{N}\}$ of states vectors $f_n \in \mathcal{D}([A, B])$ which approximates an $f \in \mathcal{D}(A) \cap \mathcal{D}(B)$ so that also the variances of A and B converge.) However, both versions (4) and (5) agree, if and only if $\mathcal{D}([A, B]) = \mathcal{D}(A) \cap \mathcal{D}(B)$. For example this is the case on the whole of $\mathcal{H}$ if both A and B are bounded.

3. The Weak CCR and the Weyl Relations

Let here A and B be two selfadjoint operators on some Hilbert space $\mathcal{H}$. A and B (are said to) satisfy the CCR (canonical commutation relations) if

$$[A, B]\,f = -\,i f; \quad \forall f \in \mathcal{D}([A, B]). \quad (6)$$

According to the notions of commutator and weak commutator, the *weak CCR* of A and B we define to be

$$\langle Af\,|\,Bg\rangle - \langle Bf\,|\,Ag\rangle = -\,i\,\langle f\,|\,g\rangle; \quad \forall f, g \in \mathcal{D}(A) \cap \mathcal{D}(B). \quad (7)$$

Obviously the weak CCR imply the usual CCR of (6). We have the following results, where (c) is added for completeness.

Theorem 2. *The following assertions are valid:*

(a) *If A and B satisfy the Weyl relations* (1), *then they fulfill the weak CCR* (7).
(b) *If A and B satisfy the weak CCR* (7), *then both, A and B, are unbounded.*
(c) *If A and B fulfill the usual CCR* (6), *then only one of the operators A and B has to be unbounded.*

Proof: (c) is proved in [5], p. 274. With Stone's theorem (a) immediately follows from

$$\begin{aligned}&\langle \exp\{-itA\}\,f\,|\,\exp\{isB\}\,g\rangle \\ &= \exp\{its\}\,\langle \exp\{-isB\}\,f\,|\,\exp\{itA\}\,g\rangle.\end{aligned}$$

(b): Let $\|f\| = 1$. Equation (4) implies $\mathrm{Var}(A, f)\,\mathrm{Var}(B, f) \geq \frac{1}{4}$. Now assume B to be bounded. Then $\mathcal{D}(B) = \mathcal{H}$ and $\mathrm{Var}(B, f) \leq 2\,\|B\|^2$. But the selfadjointness of A implies $\inf\{\mathrm{Var}(A, f); f \in \mathcal{D}(A), \|f\| = 1\} = 0$. A contradiction. □

4. The CCR in $\mathrm{L}^2([0, 2\pi], (\mathrm{d}\vartheta/2\pi))$

Here we realize the differentiating and multiplication operator in the Hilbert space $\mathcal{H}$ of square integrable functions on the intervall $[0, 2\pi]$, that is $\mathcal{H} = \mathrm{L}^2([0, 2\pi], (\mathrm{d}\vartheta/2\pi))$ with the normalized Lebesgue measure $\mathrm{d}\vartheta/2\pi$.

Before proceeding, remember that each absolutely continuous function $f\colon [0, 2\pi] \to \mathbb{C}$ is differentiable almost everywhere and its derivative f' is integrable [9], Corollary 6.3.7. Observing $\mathrm{L}^2([0, 2\pi], (\mathrm{d}\vartheta/2\pi)) \subset$

$\subset L^1([0, 2\pi], (d\vartheta/2\pi))$ let us define $\Gamma_{ac}([0, 2\pi])$ to be the set of absolutely continuous functions f on $[0, 2\pi]$ for which f' in addition is square integrable.

We define N on $\mathscr{H}$ by $Nf = -if'$ with the domain $\mathscr{D}(N) := D_0$, where

$$D_0 := \{f \in \Gamma_{ac}([0, 2\pi]);\ f(0) = f(2\pi) = 0\}.$$

Integration by parts shows N to be a symmetric operator on $\mathscr{H}$. But N is *not* essentially selfadjoint. However, N has uncountably many selfadjoint extensions N_α, which are indicated by $\alpha \in [0, 1[$ (cf. e.g. [5], p. 259)

$$\mathscr{D}(N_\alpha) := \{f \in \Gamma_{ac}([0, 2\pi]);\ f(0) = \exp\{-i2\pi\alpha\} f(2\pi)\},$$
$$N_\alpha f = -if'.$$

The spectrum of each selfadjoint N_α is purely discrete. We have for every $\alpha \in [0, 1[$

$$N_\alpha e_k^{(\alpha)} = (k+\alpha) e_k^{(\alpha)}, \quad \forall k \in \mathbb{Z},$$

where $e_k^{(\alpha)}(\vartheta) := \exp\{i(k+\alpha)\vartheta\}; \forall \vartheta \in [0, 2\pi]$. For every $\alpha \in [0, 1[$ the set $\{e_k^{(\alpha)}; k \in \mathbb{Z}\}$ forms an orthonormal basis of $\mathscr{H}$. However, $e_k^{(\alpha)} \notin \mathscr{D}(N_\beta)$ and $\mathscr{D}(N_\alpha) \cap \mathscr{D}(N_\beta) = D_0$ for any $\alpha, \beta \in [0, 1[$ with $\alpha \neq \beta$.

Let be Θ the multiplication by ϑ, $(\Theta f)(\vartheta) = \vartheta f(\vartheta)$, $\forall \vartheta \in [0, 2\pi]$, with domain $\mathscr{D}(\Theta) = \mathscr{H}$. Θ is a bounded selfadjoint operator with $\|\Theta\| = 2\pi$.

We now turn to the commutators of N_α and Θ. They satisfy the CCR (6) but *not* the weak CCR (7). Since Θ is bounded, the latter also follows from Theorem 2.

Theorem 3. *Let all be as above. The following assertions are valid for each* $\alpha \in [0, 1[$:

(a) $\mathscr{D}([N_\alpha, \Theta]) = D_0$, *with* $[N_\alpha, \Theta] f = -if; \forall f \in D_0$.

(b) *For all* $f, g \in \mathscr{D}(N_\alpha) = \mathscr{D}(N_\alpha) \cap \mathscr{D}(\Theta)$ *the weak commutator* (2) *of* N_α *and* Θ *is given by*

$$\langle N_\alpha f \mid \Theta g\rangle - \langle \Theta f \mid N_\alpha g\rangle = i\left(\overline{f(0)}\, g(0) - \langle f \mid g\rangle\right).$$

Proof: (a) $f \in \mathscr{D}([N_\alpha, \Theta])$ implies $f \in \mathscr{D}(N_\alpha)$ and $\Theta f \in \mathscr{D}(N_\alpha)$. But $(\Theta f)(0) = 0$ and hence $(\Theta f)(2\pi) = 2\pi f(2\pi) = 2\pi \exp\{i2\pi\alpha\} f(0) = 0$, and the assertion follows. (b) is easily proved with integration by parts and by observing that $\overline{f(2\pi)}\, g(2\pi) = \overline{f(0)}\, g(0)$. □

With (b) of the above Theorem and (4) we immediately obtain for the uncertainty principles for each $\alpha \in [0, 1[$

$$\mathrm{Var}(N_\alpha, f)\,\mathrm{Var}(\Theta, f)$$
$$\geq \tfrac{1}{4}\,|\langle N_\alpha f \mid \Theta f\rangle - \langle \Theta f \mid N_\alpha f\rangle|^2$$
$$= \tfrac{1}{4}\,\big||f(0)|^2 - \|f\|^2\big|^2; \quad \forall f \in \mathscr{D}(N_\alpha).$$

However, using (5) instead of (4) we get

$$\mathrm{Var}(N_\alpha, f)\,\mathrm{Var}(\Theta, f) \geq \tfrac{1}{4}\,|\langle f \mid [N_\alpha, \Theta] f\rangle|^2 = \tfrac{1}{4}\,\|f\|^4;$$
$$\forall f \in \mathscr{D}([N_\alpha, \Theta]) = D_0. \quad (9)$$

Here $\mathscr{D}([N_\alpha, \Theta]) = D_0$ is a proper subspace of $\mathscr{D}(N_\alpha)$. Especially we have $e_k^{(\alpha)} \in \mathscr{D}(N_\alpha)$ but $e_k^{(\alpha)} \notin D_0$ for any $k \in \mathbb{Z}$. Consequently, it is not allowed to use (9) for the eigenfunctions $e_k^{(\alpha)}$ of N_α.

Finally we turn to the commutation relations concerning the unitary groups generated by the operators N_α, $\alpha \in [0, 1[$, and Θ. Because of the boundedness of Θ by Theorem 2 the operators N_α and Θ do *not* fulfill the Weyl relations (1). However the following modified relations are valid.

Theorem 4. *For each* $\alpha \in [0, 1[$ *and every* $s, t \in \mathbb{R}$ *we have*

$$\exp\{itN_\alpha\} \exp\{is\Theta\}$$
$$= \exp\{its\} \exp\{is\Theta\} \exp\{itN_{(\alpha-s)\,|\,\mathrm{mod}\,1}\}.$$

Proof: Since $\exp\{it\Theta\}\, e_k^{(\alpha)} = e_k^{(\alpha+t)\,|\,\mathrm{mod}\,1}$ for every $k \in \mathbb{Z}$ the unitary $\exp\{it\Theta\}$, $t \in \mathbb{R}$, maps the orthonormal basis of $\mathscr{H}$ consisting of the eigenvectors of N_α onto the orthonormal basis of $\mathscr{H}$ consisting of the eigenvectors of $N_{(\alpha+t)\,|\,\mathrm{mod}\,1}$. We drop the "|mod 1". Then for each $k \in \mathbb{Z}$

$$\exp\{itN_\alpha\} \exp\{is\Theta\}\, e_k^{(\alpha-s)} = \exp\{itN_\alpha\}\, e_k^{(\alpha)}$$
$$= \exp\{it(k+\alpha)\}\, e_k^{(\alpha)}$$
$$= \exp\{its\} \exp\{it(k+\alpha-s)\}\, e_k^{(\alpha)}$$
$$= \exp\{its\} \exp\{is\Theta\} \exp\{it(k+\alpha-s)\}\, e_k^{(\alpha-s)}$$
$$= \exp\{its\} \exp\{is\Theta\} \exp\{itN_{(\alpha-s)}\}\, e_k^{(\alpha-s)}.$$

Since $\{e_k^{(\alpha-s)}; k \in \mathbb{Z}\}$ is an orthonormal basis of $\mathscr{H}$, the result follows.

5. Phase and Number Operators for Macroscopic Fully Coherent States

Assume $\mathscr{W}(E)$ to be the Weyl algebra over some complex pre-Hilbert space E, describing a boson system [10]. The observable phase operator as defined in [3] for representations Π of $\mathscr{W}(E)$ aims at boson systems in the thermodynamic limit, where a classical macroscopic part of the boson field occurs. The number operator N_Π is a selfadjoint operator on the representation Hilbert space $\mathscr{H}_\Pi$, which generates a unitary implementation of the gauge-transformations γ_t, which are defined to be the *-automorphisms on $\mathscr{W}(E)$ arising from the Bogoliubov transformations $\gamma_t(W(\xi)) = W(e^{it}\xi)$; $\forall \xi \in E$ (where $W(\xi)$, $\xi \in E$ are the Weyl operators), that is (cf. also [11]),

$$\exp\{itN_\Pi\}\, \Pi(T) \exp\{-itN_\Pi\} = \Pi(\gamma_t(T)); \quad \forall t \in \mathbb{R}; \quad \forall T \in \mathscr{W}(E). \tag{10}$$

An observable phase operator U is a unitary operator in the weak closure $\mathscr{M}_\Pi := \Pi(\mathscr{W}(E))''$ of the represented Weyl algebra, which satisfies

$$\exp\{itN_\Pi\}\, U \exp\{-itN_\Pi\} = \exp\{-it\}\, U; \quad \forall t \in \mathbb{R}. \tag{11}$$

With the use of the N_α of the previous section we now give an example of phase and number operators in the above sense. The example is constructed in a way similar to [3], however, using here the GNS-representation of the gauge-invariant and macroscopic fully coherent state ω_L on $\mathscr{W}(E)$ associated with an arbitrary, but unbounded linear form $L: E \to \mathbb{C}$ (unbounded with respect to the norm on E). The characteristic function of ω_L is determined by (cf. [12], [13])

$$\langle \omega_L; W(\xi) \rangle = \exp\{-\tfrac{1}{4}\|\xi\|^2\} \cdot \int_{\vartheta=0}^{2\pi} \exp\{i\sqrt{2}\,\mathrm{Re}(e^{i\vartheta} L(\xi))\} \frac{d\vartheta}{2\pi}; \quad \forall \xi \in E.$$

Its GNS representation $(\Pi_L, \mathscr{H}_L, \Omega_L)$ is given by

$$\mathscr{H}_L = \mathscr{H}_F \otimes \mathscr{K}, \quad \Omega_L = \Omega_F \otimes w,$$

$$\Pi_L(W(\xi)) = W_F(\xi) \otimes \exp\{i\sqrt{2}\,\mathrm{Re}(e^{i\vartheta} L(\xi))\},$$

where $\mathscr{H}_F$ is the Bose-Fock space over the completion of E, and $\mathscr{K}$ from Section 4 describes the mentioned classical field part, which arises since the linear form L is unbounded. Ω_F is the Fock vacuum vector and $W_F(\xi)$, $\xi \in E$, are the Fock-Weyl operators, $w(\vartheta) \equiv 1$. Moreover,

$$\mathscr{M}_L = \Pi_L(\mathscr{W}(E))'' = \mathscr{B}(\mathscr{H}_F) \otimes \mathrm{L}^\infty\left([0, 2\pi], \frac{d\vartheta}{2\pi}\right).$$

For each $\alpha \in [0, 1[$ the operator $N_\alpha^L := N_F \otimes \mathbb{1} + \mathbb{1}_F \otimes N_\alpha$, where N_F is the usual number operator on the Fock space $\mathscr{H}_F$ and N_α from Sect. 4 fulfills (10), and hence is a number operator associated with Π_L. Obviously N_α^L is not affiliated with $\mathscr{M}_L$. But $\Omega_L \in \mathscr{D}(N_\alpha^L)$, if and only if $\alpha = 0$, that is, only N_0^L is a renormalized number operator with respect to ω_L, $N_0^L \Omega_L = 0$.

Defining $U_t^L := \mathbb{1}_F \otimes \exp\{it\Theta\} \in \mathscr{Z}_L$ for every $t \in \mathbb{R}$, where $\mathscr{Z}_L = \mathscr{M}_L \cap \mathscr{M}_L'$ is the center of the von Neumann algebra $\mathscr{M}_L$, by Theorem 4 we have the relations

$$\exp\{itN_\alpha^L\}\, U_s^L \exp\{-itN_{(\alpha-s)\,|\,\mathrm{mod}\,1}^L\} = \exp\{its\}\, U_s^L; \quad \forall s, t \in \mathbb{R}.$$

Comparison with (11) yields $U_{-1}^L =: U_L$ to be an observable unitary phase operator with respect to the representation Π_L and each N_α^L.

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